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# *Structural Displacements Caused by Earthquakes: A Practical Application of Differential Equations*

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## *Abstract*

Teaching differential equations within university engineering courses allows for students to create physically realistic models whose solutions provide practical application. This work addresses the modeling of structural displacements caused by earthquakes through the use of differential equations by employing a one-degree-of-freedom bilinear oscillator model. The Euler's, Runge-Kutta and Finite Element numerical methods were used to solve the differential equation and the results were compared with simulations obtained through the DEGTRA A4 software. The capital of Oaxaca is one of the cities with the highest seismicity levels in Mexico, therefore the students from this state are well aware of the earthquake-related accelerations to which physical structures can be subject. Taking advantage of this foreknowledge, a project using differential equations was proposed, which consisted in giving numerical solutions to the model equation through diverse methods and then comparing them. The conclusions compiled in this work include students' impressions while developing the project, discussion of their results and its application in the identification of structural risk. This report suggests that this methodology was not only effective in terms of academic results but also contributed to students' professional growth.

**Keywords**—Numerical methods, structural displacement, earthquakes, differential equations.

## Resumen

La enseñanza de ecuaciones diferenciales en materias de ingeniería permite a los estudiantes modelar sistemas físicos reales, que, al darles solución, provee valores de utilidad práctica. Este trabajo aborda el modelado de los desplazamientos estructurales causados por sismos mediante ecuaciones diferenciales, utilizando un modelo de oscilador bilineal de un grado de libertad. Se emplearon los métodos numéricos de Euler, Runge-Kutta y Elemento Finito para resolver la ecuación diferencial y se compararon los resultados con simulaciones obtenidas a través del software DEGTRA A4. La ciudad de Oaxaca es una de las ciudades con más alta sismicidad de México por lo que la problemática de las estructuras sometidas a aceleraciones sísmicas es muy bien conocida entre los estudiantes de este estado. Aprovechando este contexto se planteó un proyecto que utilizara las ecuaciones diferenciales, el cual consistió en darle soluciones numéricas a la ecuación del modelo bilineal mediante diversos métodos numéricos y compararlos. Las conclusiones recopiladas en este trabajo incluyen discusión de resultados y su aplicación en la identificación de riesgo estructural causado por sismos. Esto sugiere que la metodología no sólo fue efectiva en términos de resultados académicos, sino que también contribuyó al crecimiento profesional de los estudiantes.

**Palabras claves**. Métodos numéricos, desplazamientos estructurales, sismos, ecuaciones diferenciales

## *Introduction*

High seismicity in the Oaxaca region poses a significant challenge to the structural integrity of buildings there. The last registered high-intensity seismic event occurred on September 7th, 2017 with an epicenter in the Gulf of Tehuantepec, 137 kilometers southeast of the city of Pijijiapan. The earthquake measured 8.2 Mw, greatly impacting many buildings in the region. Seismic events are frequent in the state; since its inhabitants are highly accustomed to this phenomenon, they are familiar with safety protocols and are highly aware of the effects these events have on its buildings, like those inflicted by the earthquake on September 7th 2017 in the towns in the Gulf of Tehuantepec (Pozos-Estrada et al., 2019; Godínez-Domínguez et al., 2020).

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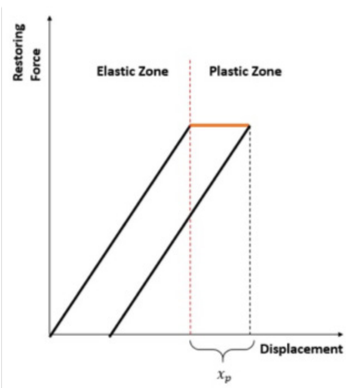
Taking this social context of into account, the familiarity with this seismic phenomenon was used to link the solution to a differential equation with a case study in which the students could identify and visualize a practical application to the solution. Applying knowledge to the use of models develops interest for application among students and gives meaning to the results, further motivating the acquisition of deeper understanding. Some concepts, abilities and mathematical processes that students develop while solving real-world problems can be found to be more meaningful (Hamilton et al., 2008).

This study aims to apply numerical modeling methods and then analyze structural movement during seismic events, focusing on a one-degree-of-freedom bilinear oscillation model that represents the loading conditions in typical structures in the region. For the reasons mentioned above, the differential equations course in the software and computer systems engineering program at La Salle University of Oaxaca (Universidad La Salle Oaxaca) planned a special exercise to solve a one-degree-of-freedom bilinear oscillation model described by a differential equation. Given that students take classes in computer programming, it was proposed that the solution methods be numerical.

The model used in this work is a one-degree-of-freedom bilinear oscillation model which describes the structural displacements on buildings caused by earthquakes ( Montuori et al., 2021; Qiu et al., 2021; Pierleoni et al., 2018).

The bilinear model "Fig. 1" consists of two zones: the linear zone that describes displacements in the elastic zone in which the structures return to their initial position, and the nonlinear one, which describes displacements in the plastic zone, in which a permanent displacement exists and the structure does not return to its original position.

Fig 1. Displacement graph – restoring force




Different models exist that describe one-degree-of-freedom oscillators. In this work, an elasto-plastic model was used, its lineal zone described by the following differential equation:

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = -\ddot{x}_g$$

Where $\ddot{x}$ is the acceleration in the oscillator, is the velocity of the oscillator, is the displacement in the oscillator, $\zeta$ and $\omega$ are parameters of the system that depend on the mass, the rigidity and the coefficient of the damping of the oscillator. $\ddot{x}_g$ is the acceleration of entry to the system, which is the acceleration in the base of the oscillator.

When solving the differential equation of this model, oscillator displacements are obtained. Theses displacements can be used to determine if the structure is found in an elastic zone or if it has passed on to its plastic zone. It is possible to model elements of a building if the structural parameters are known or approximated, if the acceleration of entry corresponds to the seismic accelerations, and if the response to the equations provides structural displacement. Several methods have been developed that can estimate structural damage (Pan and Lee, 2002; Kim and Melhem, 2004; Yang et al., 2003). In this work, structural displacement solutions have been obtained without giving interpretation to the structural damage suffered.

The Euler's and Runge-Kutta solution methods are used in this work, which were selected because they were indicated by the differential equations program. Considering the scope of the subject, the Henn or Backward Euler methods could be used, among others. These solution methods are indicated for linear differential equations, which is why only the linear zone of the bilinear oscillator model is considered. Additionally, the Finite Element Method (FEM) was incorporated because of its programming characteristics and to further incorporate a challenge for the students.

## *Methodology*

Once each of the elements related to the analytic solutions of ordinary linear differential equations had been addressed, the group proceeded to carry out examples and exercises incorporating the numerical solution methods for these types of equations with the Euler's Method, the Runge-Kutta Method and the Finite Element Method. After this process, the group discussed the earthquake predicament through a presentation focused on the consequences of earthquakes and



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the structural damages to buildings that were sustained specifically in the 2017 earthquake in the region. The professor emphasized the structural displacement buildings suffer due to seismic acceleration as well as the deformation stages that the materials suffer in the elastic or plastic zones. Once these points were covered, the model concept was introduced as a means to represent physical phenomena in order to then present the model that describes a one-degree-of-freedom bilinear oscillator, which can additionally be used to describe structural displacements caused by accelerations. Finally, the model was linked to other parts in a building and, by working in teams, students were asked to give solutions to the differential equations in three numerical methods.

In order to arrive at the numerical solutions of differential equations, it is necessary to know both the accelerations of entry to the system and that of the oscillator; both accelerations can be reduced to an absolute acceleration in the following manner:

And equation (1) is expressed as:

$$\ddot{x}_a = \ddot{x} + \ddot{x}_g \qquad (2)$$

$$2\zeta\omega\dot{x} + \omega^2 x = -\ddot{x}_a \qquad (3)$$

To solve this equation, the acceleration of entry earthquake register on September 19th, 1985 was used (also known as register “SCT”). $\zeta$ and $\omega$ were left as modifiable variables. The initial displacement was considered in the initial time of zero.

Next, we describe methods. Taking the definition of the Euler’s Method for differential equations (Ross, 2004) and applying it to (3) we obtain the following iterative expression:

$$x_i = \left(\frac{-\ddot{x}_{a_{i-1}} - \omega^2 x_{i-1}}{2\zeta\omega}\right)dt + x_{i-1} \qquad (4)$$

Here, i is the index of the actual position and i-1 is the index of the previous position. The Euler’s Method is proposed because it is a simple method to implement, although it is less precise than other methods, such as the Runge-Kutta Method.

The Runge-Kutta Method is described as (Kim and Melhem, 2004):

$$x_i = x_{i-1} + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \qquad (5)$$

Where:

$$K_1 = f(t_{i-1}, x_{i-1})dt$$

$$K_2 = f\left(t_{i-1} + \frac{1}{2}dt, x_{i-1} + \frac{1}{2}K_1\right)dt$$

$$K_3 = f\left(t_{i-1} + \frac{1}{2}dt, x_{i-1} + \frac{1}{2}K_2\right)dt$$

$$K_4 = f(t_{i-1} + dt, x_{i-1} + K_3)dt$$

Adapted to (3) the function f is represented by:

$$f(x) = \frac{-\ddot{x}_{a_{i-1}} - \omega^2 x_{i-1}}{2\zeta\omega}$$

Solving for (3) through the Finite Element Method (Zienkiewicz and Taylor, 2005) we obtain:

$$K_{ii}^e = \zeta\omega + \frac{\omega^2 T_e}{3}$$

$$K_{ij}^e = \zeta\omega + \frac{\omega^2 T_e}{6}$$

$$f_j^e = \frac{T_e \ddot{x}_a(t_i)}{2}$$

The displacements are given by the following expression:

$$[x] = K^{-1}(-f) \qquad (6)$$

Where is the inverted matrix of $K^{-1}$ which is constituted in the following manner:

$$K = \begin{bmatrix} K_{ii}^e & K_{ij}^e & 0 & 0 & \cdots & 0 \\ K_{ij}^e & 2K_{ii}^e & K_{ij}^e & 0 & \cdots & 0 \\ 0 & K_{ij}^e & \ddots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 2K_{ii}^e & K_{ij}^e & 0 \\ \vdots & \vdots & \vdots & K_{ij}^e & 2K_{ii}^e & K_{ij}^e \\ 0 & 0 & 0 & 0 & K_{ij}^e & K_{ii}^e \end{bmatrix}$$

And the vector f is formed in the following way:

$$f = \begin{pmatrix} f_j^e \\ 2f_j^e \\ \vdots \\ 2f_j^e \\ f_j^e \end{pmatrix}$$

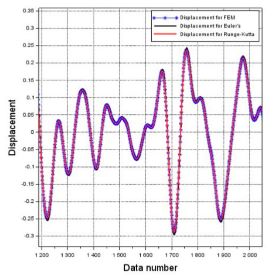
The students programmed numerical methods on Scilab, which is an open-source software for numerical computing. The equation was solved for the entry file with the values $\zeta$=0.05 and $\omega$=2π and dt=0.01, subsequently generating answer graphs. Additionally, the students used a displacement register with a bilinear oscillator simulation. This displacement register was useful to compare the solutions obtained with a simulation made by the software. The software used in this study is called “DEGTRA A4” and was developed by the National Autonomous University of Mexico (Universidad Nacional Autónoma de Mexico) which, among other functions, is a bilinear oscillation simulator. It is important to mention that the displacements obtained from the simulations were of a single bilinear oscillation, which is why they contain displacements within the linear or elastic zone and displacement within the nonlinear or plastic zone. This configuration was chosen by the professor so that students, while solving the differential equation, could conclude that numerical methods adjust well in the linear zone but not so in the nonlinear zone.

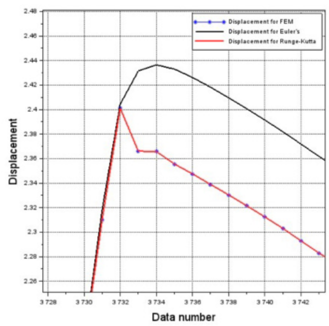
## *Results*

The students solved the numerical methods and then graphed each one in the same system coordinates for comparison. The three methods gave similar results, the difference among them is minimal, as shown in “Fig. 2”.

By comparing generated graphs to the numerical solutions, we observe an abrupt slope along the Finite Element Method data, generating a peak that the Euler’s method do not, see “Fig. 3”.

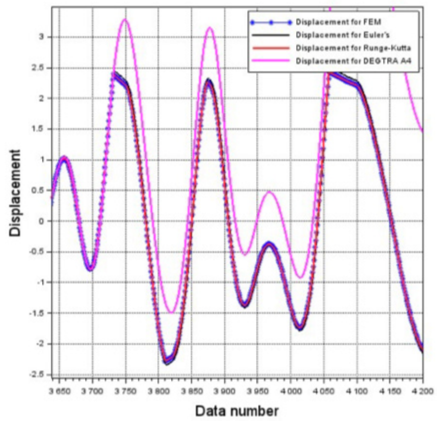
Differential equation solutions of the three numerical methods




Comparative among the three numerical methods

Additionally, the students were provided with a file of displacements generated by the DEGTRA A4 simulation software in order to compare them with the ones obtained via numerical calculation methods, see “Fig 4”. We observe that solutions obtained by the simulation software are similar to the ones obtained by numerical methods in the first data set. Notably, around data point 3731, we see a slight difference between the programmed numerical methods and the simulation.




Comparative among the three numerical methods and simulation

After having obtained all the graphs, the process of interpretation began. The first question that the students raised regarded the causation of the peaks within the Finite Element Method solutions, which students attributed to an error in the method. While conducting additional analysis of the behavior in the three numerical methods and the one simulated by software, the students observed that while the numerical method solutions followed a pattern, the ones generated by the simulator differ after the 3731 data point, but after this differing behavior, they largely coincided again with a certain offset.

The professor shared that the differences between data generated by solving differential equations and the data obtained by the simulation were due to the products of a bilinear oscillation, and that while the oscillator maintained in the linear zone, the programmed numerical methods had the same behavior. However, if the simulated answer was in the nonlinear zone, it differed with the other programmed methods; when the oscillator returned to its linear behavior, the numerical methods returned to the same behavior which is simulated but with a certain offset. This offset was due to a permanent displacement that prevented the oscillator from returning to its original position.



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The peaks that appear in the Finite Element Method were of great interest. The students noticed that the first peak began where the oscillator entered its nonlinear zone. The points where the oscillator entered its nonlinear zone through the generated register were obtained through the DEGTRA A4 software; these points were compared with the abrupt change in slope that is observed in the Finite Element Method. The following results were obtained:

Comparative between the entry to nonlinear behavior of simulated displacements and the peak of the fem

| Inputs to nonlinear behavior | | |
|---|---|---|
| Data number | Simulated displacements | Peaks of the FEM |
| | 3731 | 3731 |
| | 3810 | 3809/3810 |
| | 3874 | 3873 |
| | 4057 | 4057 |
| | 4210 | 4209 |
| | 4302 | 4301 |
| | 4407 | 4407 |
| | 4909 | 4908/4909 |
| | 5019 | 5018 |

In comparing the abrupt changes in slope in the Finite Element Method solution with the entries of nonlinear behavior in the simulated displacements, we see the data nearly coincide.

The interpretive application of this study was further revealed to the students as they recognized the capacity to estimate oscillator displacement which could represent the structural elements of a building that has sustained an earthquake. Obtaining displacement measures could indicate if a structure is within the plastic or elastic zone. Likewise, this study emphasizes the limitation of using a one-degree-of-freedom oscillator, since it only analyzes one movement of the six possible movements that a structure subject to earthquakes may sustain, as the model used in this study is a simplification of more elaborate models.

## Conclusions

As students from a highly seismic region, the students in this study were able to link the use of differential equations with familiar application. Additionally, they applied their computer knowledge to solve numerical methods. While executing this project, students expressed interest in the theory behind earthquakes and mathematical application in such events. The differences stated in the Finite Element Method generated a discussion regarding its efficacy, but after discovering that they are linked from the beginning of nonlinear behavior, one may ask whether the Finite Element Method could detect the changes in linear behavior from that of nonlinear oscillations. Thusly, in future works, we recommended conducting larger scale trials with different configurations of oscillator and accelerations to assure uniformity in all trials. We also recommended using more numerical methods to verify if other methods present the same behavior.

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